MATH 350 FINAL Fall 2004

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are <u>not permitted</u>.

1. (20 points) Let f be differentiable at every point in a closed, bounded interval $[a\ ,\ b]$. Prove that if f' is $1-1$ on $[a\ ,\ b]$, then f' is strictly monotone on $[a\ ,\ b]$.

2. (40 points) Find all $\mathcal{C}^2$ functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$\exists\ \theta \in [0\ ,\ 1]\ ,\ \forall x \in \mathbf{R} \text{ and } \forall h \in \mathbf{R}\ :\quad f(x+h)\ =\ f(x) + hf^{'}(x+\theta h)$$

3. (20 points) Let (x_n) be a sequence defined by

$$\begin{cases} x_1 \ = \ \dfrac{1}{2} \\ \\ x_{n+1} \ = \ {x_n}^2 + \dfrac{3}{16} \end{cases}$$

   a. Show that $x_n \geq 0\ ;\quad \forall n \in \mathbf{N}$

   b. Show that (x_n) is a decreasing sequence.

   c. Deduce that (x_n) is a convergent sequence and find its limit.

4. (30 points) Suppose that $f : [a\ ,\ b] \longrightarrow \mathbf{R}$ is continuous and $f(a) \neq f(b)$. Prove that if p and q are two positive real numbers, then

$$\exists\ c \in (a\ ,\ b)\ ,\quad pf(a) + qf(b)\ =\ (p+q)f(c)$$

5. (20 points) Suppose that $f : [0\ ,\ \infty) \rightarrow \mathbf{R}$ is continuous and there is l such that $\lim\limits_{x \to +\infty} f(x) = l$. Prove that f is uniformly continuous on $[0\ ,\ \infty)$.

HEY, THERE'S MORE—TURN THE PAGE OVER!

6. (20 points) Evaluate the following limits.

a. $\lim\limits_{x\to 0} \dfrac{\sin \frac{1}{x}}{e^{\frac{1}{x}}+1}$

b. $\lim\limits_{n\to\infty} \left(\dfrac{n}{n^2+1} + \dfrac{n}{n^2+2} + \dfrac{n}{n^2+3} + \cdots + \dfrac{n}{n^2+n} \right)$

c. $\lim\limits_{x\to 0} \dfrac{\sin x \log(1+x^2)}{x \tan x}$

d. $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n+1}-\sqrt{n}}$

e. $\lim\limits_{x\to\infty} x^2 \left(e^{\frac{1}{x}} - e^{\frac{1}{x+1}} \right)$

f. $\lim\limits_{x\to 0} x^n \sin \dfrac{1}{x^2}; \quad n \geq 0$

7. (30 points) Consider the sequence (x_n) such that $\forall n \in \mathbf{N}, x_n = (-1)^n$.

a. Using the definition, prove that (x_n) is a divergent sequence.

b. Use an other method to establish the divergence of the sequence (x_n).

8. (20 points) Let $f(x) = x^2 e^{x^2} \; ; \quad x \in \mathbf{R}$.

a. Show that f^{-1} exists and is differentiable on $(0 \, , \, \infty)$.

b. Compute $(f^{-1})'(e)$.